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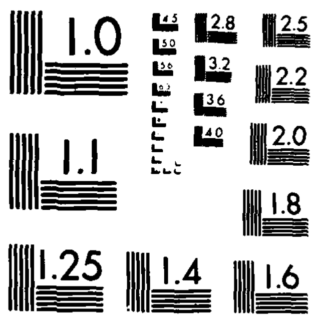

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A GEOMETRIC PROOF OF TOTAL POSITIVITY FOR SPLINE INTERPOLATION

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ABSTRACT

The possibility of expressing any B-spline as a positive combination of B-splines on a finer knot sequence is used to give a simple proof of the total positivity of the spline collocation matrix.

AMS (MOS) Subject Classifications: 41A15, 41A05, 15A48

Key Words: spline interpolation, total positivity, variation diminishing, B-polygon, adding knots

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SIGNIFICANCE AND EXPLANATION

The total positivity of the spline collocation matrix is the basis of several important results in univariate spline theory. This makes it desirable to provide as simple as possible a proof of this total positivity. The proofs available in the literature don't qualify since ~~they~~ these all rely on certain determinant identities which are not exactly intuitive. We give ~~here~~ a proof that uses nothing more than Cramer's rule (hard to avoid since total positivity is a statement about determinants) and the geometrically obvious fact that a B-spline can always be written as a positive combination of B-splines on a finer knot sequence.

The geometric intuition appealed to here stems from the area of Computer-Aided Design in which a spline is constructed and manipulated through its B-polygon, a broken line whose vertices correspond to the B-spline coefficients. If a knot is added (to provide greater potential flexibility), the new B-polygon is obtained by interpolation to the old. This has led Lane and Riesenfeld to a proof of the variation diminishing property of the spline collocation matrix and is shown here to provide a proof of the total positivity as well.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

A GEOMETRIC PROOF OF TOTAL POSITIVITY FOR SPLINE INTERPOLATION

C. de Boor[1] and R. DeVore[1,2]

§1. Introduction. Perhaps a better title would be "Adding a knot can be illuminating" since the purpose of this note is to show how this idea can be used to give simple proofs of several important properties of B-splines, including the total positivity of the B-spline collocation matrix and the sign variation diminishing property of the B-spline representation. We show that variation diminution follows immediately from the fact that a B-spline on a given grid is a non-negative linear combination of B-splines on a refined grid. We use the same fact to prove the non-negativity of any minor of the collocation matrix and, with a bit more care, even characterize which of these minors are positive.

The total positivity of the collocation matrix was originally proved by S. Karlin [5] in his development of the general theory of total positivity. Later C. de Boor gave a spline specific proof [3]. In both cases, variation diminution was derived as a consequence of total positivity. We obtain both properties directly. This was motivated in part by the work of J. Lane and R. Riesenfeld [6], who gave a direct proof of variation diminution based on spline evaluation algorithms used in computer-aided design which can be interpreted as "adding knots". But we follow Böhm's idea [1] of adding one knot at a time. We note that Jia [4] has done related work concerning the total positivity of the discrete B-spline collocation matrix.

Let $k > 0$ be a fixed integer which is the order of the splines. We call $\underline{t} := (t_i)_1^{n+k}$ a knot sequence if $t_i \le t_{i+1}$, $1 \le i < n + k$ and $t_i < t_{i+k}$, $i = 1,\dots,n$. The B-splines of order k for this knot sequence $\underline{t}$ are given by

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(1.1) $\quad N_i(x) := N_{i,\underline{t}}(x) := (t_{i+k} - t_i)[t_i,\ldots,t_{i+k}](\cdot - x)_+^{k-1}, \; i = 1,\ldots,n\,,$

where $[t_i,\ldots,t_{i+k}]$ denotes a k-th order divided difference and $u_+ := \max(u,0)$. It follows that $N_i > 0$ and supp $N_i = (t_i,t_{i+k})$. On each interval (t_j,t_{j+1}), N_i is a polynomial of order k (degree $< k$). The B-splines are linearly independent and $\sum N_j \equiv 1$ on $[t_k,t_n]$. In particular if the number $x \in (t_1,t_{n+k})$ appears exactly $k-1$ times in $\underline{t}$, then there is only one B-spline which is non-zero at x and its value at x is one. For these and other properties of B-splines, see [3].

§2. <u>Knot refinement</u>. We say that the knot sequence $\underline{s}$ is a <u>refinement</u> of $\underline{t}$ if $\underline{s}$ contains $\underline{t}$ as a subsequence. Our only tool in the subsequent arguments is the observation that

(2.1) <u>any B-spline</u> $N_j = N_{j,\underline{t}}$ <u>is a positive linear combination of some of the B-splines</u> $N_j' := N_{j,\underline{s}}$ <u>for the refined knot sequence</u> $\underline{s}$. Precisely,

$$N_j = \sum \alpha_j(i)\, N_i'$$

<u>with</u> α_j <u>nonnegative, and</u> supp $\alpha_j = [\ell,u]$, <u>where</u> (s_ℓ,s_{u+k}) <u>is the smallest segment of</u> $\underline{s}$ <u>containing</u> $(t_j,\ldots,t_{j+k})$ <u>as a subsequence</u>.

We first prove (2.1) for the special case that

$$\underline{s} = (\ \ldots,\ t_{\nu-1},\ s_\nu,\ t_\nu,\ \ldots\ )\,,$$

i.e., $\underline{s}$ is obtained from $\underline{t}$ by the addition of the knot s_ν (satisfying $t_{\nu-1} < s_\nu < t_\nu$, of course). Then

(2.2) $$N_j = \begin{cases} N_j' & \text{for } j+k < \nu \\ N_{j+1}' & \text{for } \nu < j \end{cases}.$$

For $j < \nu < j+k$, we have two ways of writing the divided difference $[s_j,\ldots,s_{j+k+1}]$:

$$\frac{S_{j+1} - S_j}{s_{j+k+1} - s_j} = [s_j,\ldots,s_{j+k+1}] = \frac{T_j - S_j}{s_{j+k+1} - s_\nu}\,,$$

with $S_i := [s_i,\ldots,s_{i+k}]$, $T_i := [t_i,\ldots,t_{i+k}]$. Therefore

$$(t_{j+k} - t_j)T_j = (s_\nu - s_j)S_j + (s_{j+k+1} - s_\nu)S_{j+1}\,,$$

hence

$$N_j = \frac{s_\nu - s_j}{s_{j+k} - s_j} N'_j + \frac{s_{j+k+1} - s_\nu}{s_{j+k+1} - s_{j+1}} N'_{j+1} , \quad j < \nu < j+k . \tag{2.3}$$

We can combine this with (2.2) into one formula, as follows:

$$N_j = (1 - \gamma_j)N'_j + \gamma_{j+1}N'_{j+1} , \quad \text{all } j , \tag{2.4a}$$

with

$$\gamma_j := \min \{ \frac{(s_{j+k}-s_\nu)_+}{s_{j+k} - s_j} , 1 \} , \quad \text{all } j . \tag{2.4b}$$

Consequently,

$$\operatorname{supp} \alpha_j = \begin{cases} [j] & , \text{ if } t_{j+k} < s_\nu \\ [j,j+1] & , \text{ if } t_j < s_\nu < t_{j+k} \\ [j+1] & , \text{ if } s_\nu < t_j \end{cases} \tag{2.5}$$

and this finishes the proof of (2.1) for this case.

The general case follows from the repeated application of this special case, by induction: Suppose that $\underline{r}$ is, in turn, a refinement of $\underline{s}$, hence

$$N'_i = \sum \alpha'_i(\ell)\, N''_\ell ,$$

with $N''_\ell := N_{\ell,\underline{r}}$. Then it follows that

$$N_j = \sum \beta_j(\ell)\, N''_\ell , \text{ with } \beta_j(\ell) = \sum_i \alpha_j(i)\alpha'_i(\ell) . \tag{2.6}$$

Therefore $\beta_j > 0$ since we already know that α_j , $\alpha'_j > 0$. Further

$$\operatorname{supp} \beta_j = \bigcup_{i \in \operatorname{supp}\alpha_j} \operatorname{supp} \alpha'_i = [\ell',u'] ,$$

with $(r_{\ell'}, \ldots, r_{u'+k})$ the smallest <u>segment</u> of $\underline{r}$ containing $(s_\ell, \ldots, s_{u+k})$ as a subsequence. But, since $[\ell,u]$ is the support of α_j , i.e., $(s_\ell, \ldots, s_{u+k})$ is the smallest segment of $\underline{s}$ containing $(t_j,\ldots,t_{j+k})$, it follows that $(r_{\ell'},\ldots,r_{u'+k})$ is also the smallest segment of $\underline{r}$ containing $(t_j,\ldots,t_{j+k})$.

The coefficient function α_j in (2.1) has been called a **discrete B-spline**. The above argument shows that the matrix $(\alpha_j(i))$ is the product of bi-diagonal matrices with nonnegative entries, hence totally positive by Cauchy-Binet. This is the basic idea behind the proof of such total positivity in Jia [4].

§3. <u>Variation diminution</u>. We use the customary notation $S^-(\alpha)$ for the number of (strong) sign changes in the sequence or function α. We want to show that $S^-(\Sigma\, \lambda_j N_j) \leq S^-(\underline{\lambda})$, i.e., the spline $f := \sum_1^n \lambda_j N_j$ changes sign no more often than its coefficient sequence $\underline{\lambda}$. This follows from:

(3.1) i) <u>if</u> $f := \sum \lambda_j N_j = \sum \lambda_j' N_j'$ <u>with</u> $N_j' := N_{j,\underline{s}}$ <u>and</u> $\underline{s}$ <u>a refinement of</u> $\underline{t}$, <u>then</u> $S^-(\underline{\lambda}) \geq S^-(\underline{\lambda}')$;

ii) <u>if, in addition,</u> $x \in (t_1, t_{n+k})$ <u>appears as a knot in</u> $\underline{s}$ <u>with (exact) multiplicity</u> $k-1$, <u>then</u> $\lambda_j' = f(x)$ <u>for some</u> j .

Property ii) is clear. To prove property i), we first consider the special case when $\underline{s}$ is obtained from $\underline{t}$ by the addition of a single knot. In that case, we infer from (2.4) that

$$\sum_1^n \lambda_j N_j = \sum_1^n \lambda_j\big((1-\gamma_j)N_j' + \gamma_{j+1}N_{j+1}'\big) .$$

Therefore

$$\sum \lambda_j N_j = \sum \lambda_j' N_j' \quad \text{with} \quad \lambda_j' := \gamma_j\lambda_{j-1} + (1-\gamma_j)\lambda_j \,, \quad \text{all } j . \tag{3.2}$$

(Here, we set $\lambda_0 := 0$.) Since $\gamma_j \in [0,1]$, this implies that $S^-(\lambda_{j-1}, \lambda_j', \lambda_j) = S^-(\lambda_{j-1}, \lambda_j)$. Therefore $S^-(\underline{\lambda}) = S^-(\ldots, \lambda_{j-1}, \lambda_j', \lambda_j, \lambda_{j+1}', \ldots) \geq S^-(\underline{\lambda}')$. This shows (3.1.i) for a single knot refinement. But then by induction we get (3.1.i) for any refinement.

Theorem 1. (Variation Diminishing Property). $S^-(\Sigma\, \lambda_j N_j) \leq S^-(\underline{\lambda})$.

<u>Proof</u>. Let $f = \sum_1^n \lambda_j N_j$. We want to show that, for any increasing real sequence $(z_i)_1^\ell$, $S^-((f(z_i))) \leq S^-(\underline{\lambda})$. We can assume that the z_i are not knots and that $z_i \in (t_1, t_{n+k})$ (since $f \equiv 0$ outside this interval). Let $\underline{s}$ be a knot refinement of $\underline{t}$ such that each z_i appears <u>exactly</u> $k-1$ times in $\underline{s}$. Then from (3.1.ii) the sequence $(f(z_i))_1^\ell$ is a subsequence of $\underline{\lambda}'$ and the desired result follows from (3.1.i). |||

It is sometimes useful to visualize the coefficients (λ_j) geometrically. If $t_j^* := (t_{j+1} + \cdots + t_{j+k-1})/(k-1)$, then the continuous piecewise linear function $P(f,\underline{t})$ with vertices (t_j^*, λ_j), $j = 1,\ldots,n$ is called the B-polygon of f. This polygon

changes sign exactly as often as $\underline{\lambda}$. For a single knot refinement $\underline{s}$ of $\underline{t}$, the points s_j^* are related to t_j^* as in (3.2), i.e.,

$$s_j^* = \gamma_j t_{j-1}^* + (1-\gamma_j)t_j^* .$$

Hence the vertices of $P(f,\underline{s})$ lie on $P(f,\underline{t})$; which is another way of viewing property (3.1.i).

§4. Spline interpolation. We now consider spline interpolation at nodes $(x_i)_1^n$, $x_1 < x_2 < \cdots < x_n$ (later we allow coalescence). Given $(y_i)_1^n$, we have the interpolation problem

$$\sum_{j=1}^{n} \lambda_j N_j(x_i) = y_i, \quad i = 1,\ldots,n \tag{4.1}$$

with coefficient matrix

$$A := A_{\underline{t}} := (N_j(x_i))_{i,j=1}^n . \tag{4.2}$$

In case $x_i = t_j$, we require that this point appear at most a total of k times in $\underline{x}$ and $\underline{t}$.

We will show that A is totally positive and furthermore characterize which minors of A are strictly positive. For this, let B be a square submatrix of A,

$$B = A(I,J) := \left(N_j(x_i)\right)_{i\in I, j\in J} ,$$

with I and J subsequences of $(1,2,\ldots,n)$ of the same length,

$$I =: (i_1, \ldots, i_m) , \quad J =: (j_1, \ldots, j_m) ,$$

say. We call such a submatrix "good" if all its diagonal entries are nonzero. This is a natural distinction to make here because

(4.3) <u>if</u> B <u>is not "good", then</u> $\det B = 0$.

Indeed, assume that $N_{j_p}(x_{i_p}) = 0$ for some p. Then $x_{i_p} \notin (t_{j_p}, t_{j_p+k})$. Assume that $x_{i_p} < t_{j_p}$. Then $N_j(x_q) = 0$ for $q < i_p$, $j > j_p$, and this shows that columns p, ..., m of B have nonzero entries only in rows $p+1$, ..., m, hence are linearly dependent. So, $\det B = 0$. The argument for the case $x_{i_p} > t_{j_p+k}$ is similar.

Next, we write $\det B$ as a linear combination of determinants of the form $A'(I,K)$ with

$$A' := \left(N'_j(x_i)\right)$$

and (N'_j) the B-splines for a refinement $\underline{s}$ of $\underline{t}$. Precisely, we claim that, for a certain nonnegative a_J,

$$\text{(4.4a)} \qquad \det A(I,J) = \textstyle\sum^{+} a_J(K) \det A'(I,K)$$

with the superscript "+" indicating that the sum is only over increasing K. Further,

$$\text{(4.4b)} \qquad \operatorname{supp} a_J = \operatorname{supp} \alpha_J ,$$

where $\alpha_J(K) := \alpha_{j_1}(k_1) \cdots \alpha_{j_m}(k_m)$ and the α_j are as in (2.1).

For the proof, we consider first the special case that $\underline{s}$ is obtained from $\underline{t}$ by the addition of a single knot. Since $N_j = \sum \alpha_j(i) N'_i$ by (2.1), the linearity of the determinant as a function of the columns gives

$$\text{(4.5)} \qquad \det A(I,J) = \textstyle\sum \alpha_J(K) \det A'(I,K)$$

with $\alpha_J(K) := \alpha_{j_1}(k_1) \cdots \alpha_{j_m}(k_m)$. Recall from (2.5) that $\operatorname{supp} \alpha_j \subseteq [j, j+1]$. Therefore, retaining in (4.5) only terms with $\alpha_J(K) \neq 0$, we have $k_p = j_p$ or j_p+1, all p. Thus K is strictly increasing unless $k_p = k_{p+1}$ for some p (possible in case $j_p+1 = j_{p+1}$). But in the latter case, the determinant is trivially zero and hence can be ignored. This finishes the proof of (4.4) for this special case.

We prove the general case by induction on the length difference $d := |\underline{s}| - |\underline{t}|$, having just proved it for $d = 1$. Assuming it correct for a given d, let $\underline{r}$ be a refinement of $\underline{t}$ with $|\underline{r}| - |\underline{t}| = d+1$ and let $\underline{s}$ be a one-point refinement of $|\underline{t}|$ which is refined by $\underline{r}$. Then, with

$$A'' := \left(N''_j(x_i)\right) \quad \text{and} \quad N''_j := N_{j,\underline{r}}, \text{ all } j ,$$

we have $N'_j = \Sigma\, \alpha'_j(\ell)\, N''_\ell$. Further, from (4.5) and the induction hypothesis,

$$\det A(I,J) = \textstyle\sum^{+} b_J(L) \det A''(I,L)$$

with

$$\text{(4.6)} \qquad b_J(L) := \textstyle\sum^{+} \alpha_J(K) a'_K(L) > 0 ,$$

which makes (4.4a) obvious.

The proof of (4.4b) is a bit more complicated. It can be skipped if only the total positivity of A is of interest. We must show that $\operatorname{supp} b_J = \operatorname{supp} \beta_J$, with $\beta_J(L) := \beta_{j_1}(\ell_1) \cdots \beta_{j_m}(\ell_m)$. Suppose first that $\beta_J(L) = 0$. Then $\beta_j(\ell) = 0$ for some $j \in J$, $\ell \in L$. Therefore, from (2.6), $\Sigma\, \alpha_j(i)\, \alpha_i'(\ell) = 0$, and, since all terms in this sum are nonnegative, they must all be zero. Thus, $\alpha_J(K)\alpha_K'(L) = 0$ for all K. But by induction hypothesis, $\operatorname{supp} a_K' = \operatorname{supp} \alpha_K'$, therefore also $\alpha_J(K)\, a_K'(L) = 0$ for all K. We conclude with (4.6) that $\operatorname{supp} b_J \subset \operatorname{supp} \beta_J$.

To see that $\operatorname{supp} b_J \supset \operatorname{supp} \beta_J$, we must show that

(4.7) $\qquad \beta_J(L) \neq 0$ implies $\alpha_J(K)\, \alpha_K'(L) \neq 0$ for some $\underline{\text{increasing}}$ K.

Since $\operatorname{supp} a_K' = \operatorname{supp} \alpha_K$, this implies that $\alpha(K)a_K'(L) \neq 0$ for this increasing K, hence also $b_J(L) \neq 0$ from (4.6).

For the proof of (4.7), it is sufficient to show the existence of a K with

(4.8) $\qquad k_p \in A_{j_p} := \{i : \alpha_{j_p}(i)\alpha_i'(\ell_p) \neq 0\}$, all p,

and $k_p < k_{p+1}$, all p. Since

$$\beta_j(\ell) = \alpha_j(j)\alpha_j'(\ell) + \alpha_j(j+1)\alpha_{j+1}'(\ell),$$

$\beta_J(L) \neq 0$ implies that

$$\emptyset \neq A_j \subseteq \{j, j+1\}, \quad \text{all } j \in J.$$

Hence, the existence of K satisfying (4.8) is assured. To finish the proof, we must show that it is possible to choose such a K which is also increasing. If $A_{j_p} \cap A_{j_{p+1}} = \emptyset$, then we have $k_p < k_{p+1}$ for any K satisfying (4.8). Thus we only have to consider how to choose the components of K corresponding to a **connected component** $A_{j_p}, \ldots, A_{j_q}$. By this we mean that

$$A_{j_\nu} \cap A_{j_{\nu+1}} \neq \emptyset \quad \text{for } p \le \nu < q,$$

while, for any $i \neq j_p, \ldots, j_q$,

$$A_i \cap A_{j_\nu} = \emptyset \quad \text{for } p \le \nu \le q.$$

Then we can write $(j_p, \ldots, j_q) = (j, j+1, \ldots, j')$, hence, $q-p = j'-j$. Further, $i \in A_i$ for $i = j+1, \ldots, j'$. Hence, if also $j \in A_j$, then the choice $k_\nu = j_\nu$, all ν, will do. In the same way, we have $i+1 \in A_i$ for $i = j, \ldots, j'-1$. Hence, if $j'+1 \in A_{j'}$,

then the choice $k_\nu = j_\nu+1$, all ν , will do. We claim that the remaining case

$$j \notin A_j \quad \text{and} \quad j'+1 \notin A_{j'}$$

cannot occur since it would imply that there are at least k entries in $\underline{r}$ between r_{ℓ_p} and r_{ℓ_p+k} . Indeed, with supp $\alpha_j^j =: [\ell,u]$, it would follow that $u < \ell_p$, while also $\ell_q < \ell'$, with supp $\alpha_{j'+1}^j =: [\ell',u']$. Further, let s_ν be the additional knot in $\underline{s}$. Then, by (2.5), $A_i \cap A_{i+1} \neq \emptyset$ implies supp $\alpha_i = \{i,i+1\}$, hence, by (2.5), $s_i < s_\nu < s_{i+k}$, $i=j,\ldots,j'$, therefore $s_{j'+1} < s_{j+k}$, and so $\ell' < u+k$ while also $p+k-q-1 = j+k-(j'+1) \leq u+k-\ell'$. This would imply that

$$\ell_p < \ell_{p+1} < \ldots < \ell_q < \ell' < u+k < \ell_p+k \ ,$$

hence $k = \ell_p+k - \ell_p \geq 1 + (u+k-\ell') + 1 + q-p \geq 1 + (p+k-q-1) + 1 + q-p = k+1$.

Theorem 2. <u>The matrix A of (4.2) is totally positi</u>[ve]. <u>Moreover, the submatrix B of A formed by rows $i_1,\ldots,i_m$ and columns $j_1,\ldots,j$ [$_m$] has a positive determinant if and only if it is "good", i.e.,</u>

$$x_{i_\nu} \in \text{supp } N_{j_\nu}, \quad \nu = 1,\ldots,m$$

<u>Proof</u>. We already proved that $\det B = 0$ unless B is "good". Now, to prove that a "good" B has a positive determinant, we choose a refinement $\underline{s}$ of $\underline{t}$ so fine that

(4.9) <u>for each</u> $i \in I$, $N_j^j(x_i) \neq 0$ <u>implies that</u> $N_j^j(x_p) = 0$ <u>for all</u> $p \neq i$.

Then each $A'(I,K)$ appearing in (4.4a) has at most one nonzero entry in each column, hence is "good", therefore nonzero, only if it is diagonal, in which case its determinant is obviously positive. To finish the proof, we must show that at least one of the matrices appearing in the sum in (4.4a) with a positive coefficient is "good". Here is one such. Choose K so that s_{k_p} is the first point in $\underline{s}$ to the left of x_{i_p} , $p=1,\ldots,m$. Since $N_{j_p}(x_{i_p}) \neq 0$, this implies that $\alpha_{j_p}(k_p) \neq 0$, all p . |||

Corollary. (I. Schoenberg and A. Whitney [7]). <u>The interpolation problem</u> (4.1) <u>has a unique solution for all</u> $(y_i)_1^n$ <u>if and only if</u> $x_i \in \text{supp } N_i$, $i = 1,\ldots,n$.

We can also allow coalescence of the interpolation nodes. If (z_i) is such a nondecreasing sequence of nodes, then we can think of it as the limit of strictly increasing sequences (x_i). Correspondingly, repetition of a z_i corresponds to repeated or osculatory interpolation, i.e., the matching of higher derivatives. Precisely, (4.1) becomes

$$(4.10) \qquad \sum_1^n \lambda_j D^{\mu_i} N_j(z_i) = y_i, \qquad i = 1,\dots,n$$

where μ_i is the number of $j < i$ for which $z_j = z_i$. We still require that any point appear at most k times totally in $\underline{z}$ and $\underline{t}$. The coefficient matrix of (4.7) is

$$(4.11) \qquad A := A_{\underline{t},\underline{z}} := (D^{\mu_i} N_j(z_i))_{i,j=1}^n .$$

It is clear that A need not be totally positive since entries involving derivatives may be negative. However, as a well-known argument shows, if M is a minor formed by rows $i_1,\dots,i_m$ and columns $j_1,\dots,j_m$ with the property

$$(4.12) \qquad i_{\nu-1} < i_\nu - 1 \text{ implies } z_{i_\nu - 1} < z_{i_\nu}, \quad \nu = 1,\dots,m ,$$

then $M > 0$. In fact, if $M(\underline{x})$ denotes a minor corresponding to distinct nodes $\underline{x} = (x_1,\dots,x_m)$, then subtracting row one from row two shows that $M(\underline{x})/(x_2 - x_1)$ converges as $x_2 \to x_1$ to the minor M' which replaces row two of $M(\underline{x})$ by first derivatives at x_1. Hence $M' > 0$. Using this type of limiting process we see that any minor M satisfying (4.12) is non-negative.

We can also characterize those M satisfying (4.12) which are positive, namely, they satisfy

$$(4.13) \qquad z_{i_\nu} \in \operatorname{supp} N_{j_\nu}, \quad \nu = 1,\dots,m .$$

The necessity of (4.13) is proved in the same way that the necessity of (4.9) was established.

The sufficiency of (4.13) is proved by making slight modifications to the earlier proof. For this, it will be convenient to allow a point z_i to appear a total of more

than k times in $\underline{s}$ and $\underline{x}$. This is acceptable provided we stipulate that all B-splines and their derivatives be interpreted as right limits at such z_i, that is at z_i^+. With this, let $\underline{s}$ be a refinement of $\underline{t}$ such that each node z_i appears as a knot in $\underline{s}$ exactly k times, and similarly each t_i appears in $\underline{s}$ exactly k times. If J satisfies (4.13), we choose L so that $s_{\ell_p} = z_{i_p}$ and the number of $j < \ell_p$ with $s_j = s_{\ell_p}$ is μ_{i_p}. Since the coefficients $\alpha(K)$ in (4.4a) are independent of $\underline{x}$, we then obtain det $A(I,J)$ as a positive combination of certain (nonnegative) minors of A' $:= A_{\underline{s},\underline{z}}$. In particular, the submatrix $A'(J,L)$ will appear in that sum with positive coefficient since $\alpha_J(L) > 0$, and det $A'(J,L) > 0$ since $A'(J,L)$ is lower triangular with positive diagonal. We have therefore proved the following theorem.

Theorem 3. For the matrix A of (4.11), and each I, J satisfying (4.12), det $A(I,J) \geq 0$. This minor is positive if and only if (4.13) is satisfied. In particular (4.10) has a unique solution if and only if $z_i \in$ supp N_i, $i = 1,\ldots,n$.

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The possibility of expressing any B-spline as a positive combination of B-splines on a finer knot sequence is used to give a simple proof of the total positivity of the spline collocation matrix.

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